Jacek Dziubański
Uniwersytet Wrocławski, Poland

Harmonic analysis and function spaces associated with Dunkl operators

Dunkl operators

$$T_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha\in R} \frac{k(\alpha)}{2}\langle \alpha, \xi\rangle \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle}$$

on a Euclidean space $(\mathbb{R}^N, \langle\cdot,\cdot\rangle)$ are perturbations of the classical directional derivatives ∂_ξ by difference operators associated with reflections related to a root system R. Here

$$\sigma_\alpha x = x - 2\frac{\langle \alpha, x\rangle}{\|\alpha\|^2}\alpha$$

denotes the reflection with respect to the hyperplane $\alpha^\perp$ perpendicular to the root α and $k(\alpha)$ is a non-negative function which satisfies $k(\sigma_\alpha \beta) = k(\beta)$, $\alpha, \beta \in R$. They were introduced by C.F. Dunkl and have applications in mathematical physics. Harmonic analysis associated with the Dunkl operators is a generalization of the classical one. A main object of the theory is the Fourier-Dunkl transform

$$\mathcal{F}f(\xi) = c_k \int_{\mathbb{R}^N} E(-i\xi, x) f(x)\, w(x) dx,$$

where $w(x) = \prod_{\alpha\in R} |\langle \alpha, x\rangle|^{k(\alpha)}$ and $E(i\xi, x)$ is the so-called Dunkl kernel (a generalization of the exponential function $e^{i\langle \xi, x\rangle}$). The Dunkl transform is an isometric transform on $L^2 = L^2(w(x)dx)$. In analogue of the classical Fourier analysis, the generalized translations and convolutions are defined by the formulae

$$\tau_x f(y) = \mathcal{F}^{-1}(\mathcal{F}f(\cdot)E(ix, \cdot))(y), \quad f * g = \mathcal{F}^{-1}(\mathcal{F}f\mathcal{F}g), \quad f, g \in \mathcal{S}(\mathbb{R}^N).$$

Boundedness of the translations on L^p-spaces as well as the Young inequality $\|f * g\|_{L^p} \leq C\|f\|_{L^1}\|g\|_{L^p}$ are open problems in the theory. The last is known to hold if $p = 2$ or one of the functions is radial. The Dunkl-Laplace operator $\Delta_k = \sum_{j=1}^N T_{e_j}$ is a generator of a semigroup $e^{t\Delta_k} f(x) = f * h_t(x) = \int f(y)h_t(x, y)\, w(x)dx$ of contractions on $L^p(w(x)dx)$. One can ask about the following topics in the theory of the Dunkl operators:

- properties of the generalized translations $\tau_x f$
- upper and lower estimates for the Dunkl heat kernel $h_t(x, y)$
- boundedness of multiplier operators $f \mapsto \mathcal{F}^{-1}(m(\xi)\mathcal{F}(\xi))$ on function spaces
- properties of Δ_k-harmonic functions
- possible characterizations of Hardy spaces (by: maximal functions, relevant Riesz transforms, atomic decompositions, square functions)
- properties of BMO_k functions

During the talk we shall present selected result of the theory. These are joint works with Jean-Philippe Anker and Agnieszka Hejna.

References.

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